Three-dimensional Iroshnikov-Kraichnan turbulence in a mean magnetic field

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Abstract

The Iroshnikov-Kraichnan cascade has been proposed in 1965-67 as a weak, quasi-isotropic turbulence theory that extended the Kolmogorov cascade to MHD. When a strong mean field is present, the more recent (Goldreich-Sridhar 1995) theory does not eliminate the possibility of weak turbulence in MHD (see Galtier et al 2000), but forces the cascade to be quasi-perpendicular. We show that other possibilities exist, which leads to a revival of IK cascade in a 3D version with a mild anisotropy.

Introduction

When a strong mean magnetic field B° is present, incompressible MHD turbulence produces small scales mainly in directions perpendicular to the mean field, because only in these directions will non-linear couplings be resonant.

The classical scenario (since Goldreich & Sridhar 1995 - GS) predicts in this case a *strong* cascade with a 1D energy spectrum scaling as $E(k_\perp) \propto k_\perp^{-5/3}$, with possibly at large scales a $k_\perp^{-2}$ scaling, characteristic of a weak cascade, if large enough Alfvén waves with parallel waveverctors are present, thus leading to a weak turbulent regime with non-resonant couplings in this large-scale range.

Recently, we have analyzed a different regime (Müller Grappin 2005, Grappin Müller 2010 - GM10) in which the cascade is much more isotropic, that is, a $k^{-3/2}$ scaling is found in all directions of Fourier space, with however a strong anisotropy in amplitude, thus making a bridge between the isotropic weak regime of Iroshnikov (1965) and Kraichnan (1967) (IK) and the classical perpendicular GS cascade.

Isotropic and anisotropic Alfvén waves turbulence

The sketch below represents the first theory of turbulence driven by Alfvén waves coupling. Coupling occurs only between waves with opposite cross-helicity ($z^+ = u+b$ and $z^- = u-b$ fields).

These waves propagate fast in opposite directions (we neglect those with wavenumbers perpendicular to the mean field), so many successive interactions are necessary for energy to cascade.

The *effective* energy transfer time is thus larger than the Kolmogorov nonlinear time $t_{NL} = 1/(ku)$, where k is wavenumber and u the amplitude at the scale 1/k.

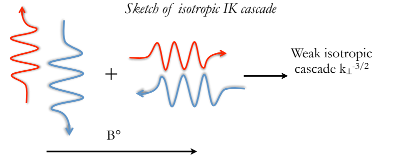
The effective transfer time is $t^* = t_{NL}\,(B°/u)$. The cascade is thus characterized by an energy flux

$F = u^2/t^* = ku^4/B°$

which finally leads to $u \approx k^{-1/4}$ hence the energy spectrum $E(k) \approx u/k$:

$E(k) \approx k^{-3/2}$

Sketch of isotropic IK cascade



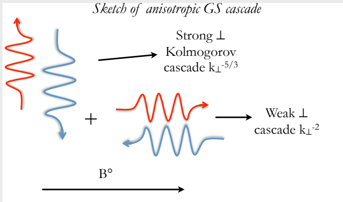
The sketch below represents the modern (GS) theory of MHD turbulence with mean field, designed to take into account anisotropy wrt mean field direction. Two situations are considered:

(i) substantial amplitude of large-scale parallel Alfvén waves ⇒ the cascade is weak (but purely perpendicular)

(ii) no such Alfvén waves ⇒ the cascade is the strong Kolmogorov one.

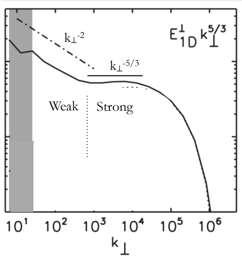
The mean field here has the sole effect of constraining the cascade to be perpendicular.

Sketch of anisotropic GS cascade



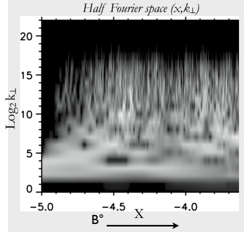
GS perpendicular cascade: the multi-shell example

1D total perpendicular energy spectrum



The two successive weak/strong cascades with the -2 and -5/3 1D scaling for the total perpendicular energy spectrum (kinetic +magnetic) have been obtained here via a *multi-shell* model of reduced MHD, allowing to reach a Reynolds number ≈ 10^6, with 10^4 grid points in the direction parallel to the mean field and a perpendicular resolution of 2^{20}, thus 20 modes in each of the 10^4 perpendicular planes supporting the perpendicular cascade (Verdini Grappin, 2012).

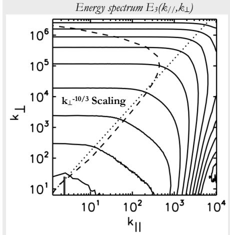
Half Fourier space $(x, k_\perp)$



Mean field is $B°/b_{rms}=5$, forcing is isotropic on $1 \leq k \leq 2$, with short correlation time.

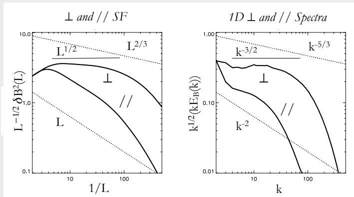
Turbulence is characterized by a *limited parallel extension* of the spectrum, given by the large-scale conditions for the weak regime. On the left we show a snapshot solution in half Fourier space. The parallel width of the high-$k_\perp$ tails is $\Delta X \approx 0.01$, while the corresponding ⊥ width is $1/k_\perp \approx 2^{-17}$. In this regime *critical balance* between the non-linear and Alfvén times is satisfied: $k_{//} \approx k_\perp^{2/3}$.

Energy spectrum $E_3(k_{//},k_\perp)$



Isocontours of the 3D energy spectrum in full Fourier space $E_3(k_{//},k_\perp)$ (on the left) show that the main scaling is perpendicular (the reduced 1D spectrum $\propto k_\perp^{-5/3}$), with secondary scalings in the parallel direction (see Verdini Grappin, 2012).

GM10 $k^{-3/2}$ regime: 3D MHD simulation



The simulation has mean field B°/brms=5, with isotropic *forcing* on $1 \leq k \leq 2$, with infinite correlation time (frozen fields. Figure on the left shows 1D spectra and structure functions (SF)

$SF(\mathbf{L}) - = \delta B^2(\mathbf{L}) = \langle (B(\mathbf{x}+\mathbf{L}) - B(\mathbf{x}))^2 \rangle$

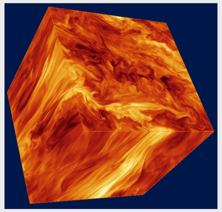
with **L** // or ⊥ to B°. They are *strongly anisotropic*:

$SF \propto L_\perp^{1/2}$ and $\propto L_{//}$ (Müller et al 2003). Spectra $E(k) \propto k_\perp^{-5/3}$

In principle, one should have

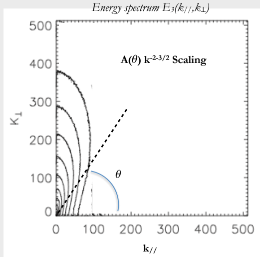
$SF(1/k) \approx kE(k)$ (1)

hence $E(k_{//}) \approx k_{//}^{-2}$; $E(k_\perp) \approx k_\perp^{-3/2}$



A snapshot in real space is shown on the left. The magnitude of the magnetic field is shown, revealing the much longer typical wavelength along the field than across it.

Energy spectrum $E_3(k_{//},k_\perp)$



The figure below (GM10) shows the 3D total energy spectrum, which appears to produce a single scaling in all directions:

$E_3(k,\theta) \approx A(\theta)\, k^{-2-3/2}$

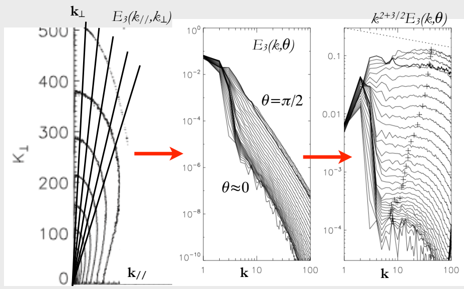
This is found by making radial cuts at different angles θ wrt the mean field direction.

The three panels on the bottom figure show the method.

left: isocontours of the 3D spectrum with radial lines at different angles $\theta \in [0, \pi/2]$;

middle: the resulting spectral densities $E_3(k,\theta)$ vs k for different angles

right: same but compensated by $k^{2+3/2}$. (Dotted: $k^{-2-5/3}$).



Remarks:

(i) the power-law range extends at most up to k=50 (see crosses in the right panel that mark the beginning of the dissipation range)

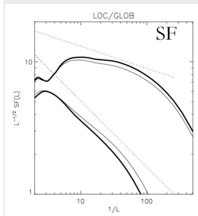
(ii) A 3D index ≈ -(2+3/2) in all directions corresponds to a -3/2 1D index

(iii) the power-law range reduces as direction becomes more parallel

(iv) Scaling cannot be obtained for $\theta \leq 10°$ due to evident lack of statistics

GM10: isotropic or anisotropic scaling?

Local and global frames give the same SF



A solution to the apparent contradiction between the anisotropic scalings obtained vis SF and the isotropic scaling of the 3D spectrum consists in remarking that the SF are measured with respect to directions parallel or ⊥ to the *local mean magnetic field*, while the 3D spectrum is measured in an *absolute frame*, thus different from the previous one.

However, this hypothesis is wrong, at least for the present value of the ratio $B°/b_{rms} = 5$, as is shown in figure on the left.

The *local* SF have thick lines, the *global* SF have thin lines. The scaling laws are basically the same in both cases.

From 3D to 1D

Since, as we know now, SF measured in local or global frames are equivalent, we adopt the global frame. In this frame, there are two different ways to measure SF: either directly in real space, or indirectly by deducing it from the 3D spectrum. The latter may be done as follows.

(1) We deduce the 1D spectrum from the 3D spectrum by integrating on perpendicular directions the 3D spectrum.

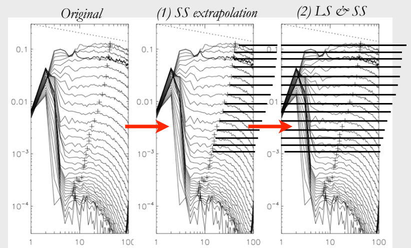
(2) We deduce the SF from the 1D spectrum by using the equality (E = total energy, E(k) = 1D reduced spectrum, F^{-1} the inverse Fourier transform):

$SF(L) = 2(E - F^{-1}(E(k))$ (2)

The hypothesis we will test now is that the 3D → 1D reduction transforms the 3D strong amplitude anisotropy into a spectral slope anisotropy.

There are indeed important non-scaling ranges in the 3D spectrum, which may be responsible of that.

In the following, we will thus try to progressively eliminate the non-scaling parts of the 3D spectrum and compute the associated SF and 1D spectra.



Above we sketch the two transformations we want to apply to the 3D spectrum (left panel shows the original 3D spectrum).

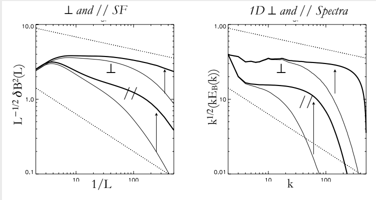
(1) Extrapolating small scales (*SS*)

We replace the dissipative tail by an exact $k^{-2-3/2}$ law, starting at the start of the dissipation range.

(2) Extrapolating large and small scales (*LS and SS*)

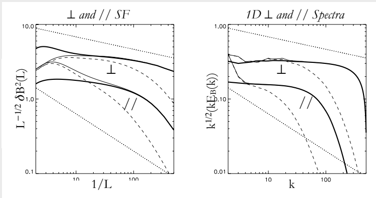
We replace the whole spectrum by an exact $k^{-2-3/2}$ law, taking the amplitude at the start of the dissipative tail as a reference.

1. SS extrapolation



Above we show the effect of extrapolating small scales. The arrows show the respective effect on the parallel and perpendicular directions, both for SF (left) and 1D spectra (right).
In both cases, the slopes become flatter, and the difference between the perpendicular and parallel scaling is much reduced.
The spectrum in particular shows an almost 3/2 scaling in the ⊥ as well as the // direction.
However, the SF anisotropy is still large. This difference between the SF and 1D spectral scalings comes from the fact that the relation (1) is valid only in the limit where a single power-law holds (either for SF or SP), which is clearly not the case here, in view of the important large-scale non-scaling ranges.

2. LS and SS extrapolation



When both large and small scales are replaced by the ideal $k^{-2-3/2}$ scaling, the anisotropy between the parallel and perpendicular directions disappear for both the SF and SP.
We conclude that the finite extent of the scaling laws is responsible for the observed differences in 3D and 1D properties.

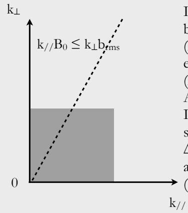
GM10 phenomenology

We propose here a phenomenology of the GM10 cascade that will describe its main properties. This will be done in several steps. Note that our ambition is not to describe the detailed variation of amplitude of the spectrum in the radial power-law range, nor to describe the non-scaling properties (in particular the transition between the isotropic forced scales and the power-law range).
We instead propose two couples cascades, one in the perpendicular directions, the other one summarizing the cascade in the non-perpendicular directions, including the parallel one.

On the nature of the ⊥ cascade

We propose that the ⊥ cascade is a 2D version of the isotropic weak IK cascade, based on the rms field amplitude b_{rms} (not B°).
Basically, this is the sole hypothesis which allows to recover many observed properties of the GM10 regime:
1. 1D $k^{-3/2}$ scaling law for total energy
2. k^{-2} scaling for residual energy (magnetic - kinetic) (Müller Grappin 2005).
3. Correlation time for the signal at scale $1/k_\perp$ based on the ⊥ Alfvén period $t_A = 1/(k_\perp b_{rms})$.
Note that in the opposite case of a strong cascade the correlation time is equal to the nonlinear time $t_{NL} = 1/(k_\perp u) \propto k_\perp^{-2/3}$, as found in multi-shell simulations by Verdini Grappin (2012).

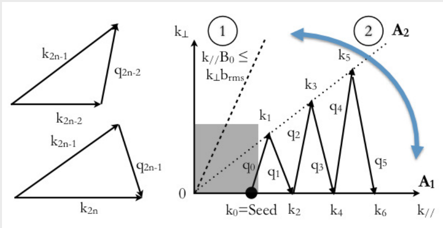
Critical balance applied to the ⊥ IK cascade



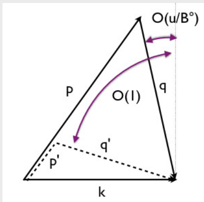
In the case of the IK cascade, the critical balance argument is expressed as follows.
(i) The correlation time of perpendicular eddies is $1/(kb_{rms})$
(ii) it is equal to the Alfvén period of parallel Alfvén waves
In conclusion, the parallel width of the spectrum should satisfy
$\Delta k_{//} = k_\perp b_{rms}/B°$
as sketched in the left figure.
(Gray area denotes the forced area).

The ricochet process

We assume that the GM10 regime relies on *quasi-local* interacting triads (k,p,q) that is $k \approx p \approx q$. The triads are such as to allow propagation along oblique lines crossing the origin (see lines A1 and A2 in the right panel below), in the region outside the critical balance region (1).
The signal is propagating from k_n to k_{n+1}, switching from line A1 to line A2 and back to A1 etcc... , hence the name ricochet.
The triads that drive the ricochet process are represented in the left panel.
Wavevectors q_n correspond to quasi-perpendicular modes (see below); the reservoir for these modes is found within the area labeled "1" that is bounded by the critical balance area for the perpendicular IK cascade.



Quasi-resonant condition, flux reduction and oblique scaling



We ask the coupling to be quasi-resonant, which imposes to retain only triads veryfing $t_{NL} \leq t_A$, which reads
$q_x B° \leq q\, u_q$
This implies (see figure on the left) that only a fraction R of the triads is to be retained, with
$R = q_x/q = u_q/B°$

Finally, this implies that the nonlinear energy flux F flowing along the oblique lines be reduced by the same factor R:
$F = u_k^2/t_{NL} = ku_q\, u_k^2 \rightarrow ku_q\, u_k^2\, (u_q/B°) = ku_q^2\, u_k^2\,/B°$

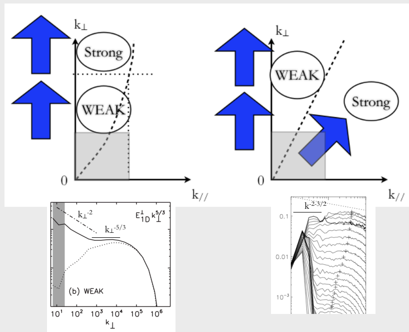
In a stationary cascade, the flux is scale-independent (F = constant); then, replacing u_q by the *perpendicular* IK scaling
$u_q \approx q^{-1/4}$
in the flux expression, we obtain
$u_k \approx k^{-1/4}$
which implies the IK scaling for the 1D reduced spectra in oblique directions $E(k) \propto k^{-3/2}$, and a 3D spectrum scaling as $k^{-2-3/2}$.

Spectral aspect ratio

We can recover one of the main property characterizing the anisotropy of the 3D GM10 spectrum, namely the ratio between the // and ⊥ inertial ranges. This is done by balancing the energy flux respectively in the ⊥ and oblique cascade; we obtain successively in the ⊥ and // directions:
$\nu k^2 = ku_q^2/b_{rms}$
$\nu k^2 \approx ku_q^2/B°$
which leads to the aspect ratio
$(k_d)_{//}/(k_d)_\perp = b_{rms}/B°$

Conclusion

(1) Standard ⊥ cascade (2) 3D IK cascade

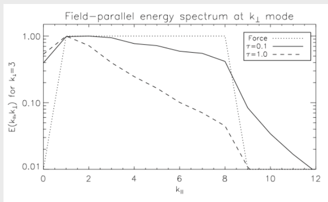


The two regimes are summarized above. The gray region represents large-scale forcing.
1. Left, we represent the classical perpendicular cascade, with the weak cascade at the largest scales, the strong cascade at smaller scales and a mild parallel extension of the spectrum.
2. Right, we represent the GM10 regime, in which the weak (2D isotropic) regime based holds in the perpendicular direction, with the oblique and parallel directions being excited by a strong cascade enslaved to the quasi-perpendicular IK cascade. We propose to call this true 3D cascade the IK cascade.

Note small-scale cross-helicity scaling (see Boldyrev, 2005) is not discussed here. The GM10 regime actually shows some small-scale cross-helicity scaling, that might play a role in the dynamics; we refer the reader to Grappin et al 2013 for a detailed discussion of this issue.

Conjecture

The two regimes, GS and GM10, are found when isotropic forcing is used at large scales. We conjecture here that the bifurcation is controled by the properties of large-scale forcing., with the first regime resulting from forcing with short correlation time, and the 3D IK regime resulting from forcing with long correlation time.
We indeed found indications in this direction by studying the multishell system (Verdini Grappin 2012).
Published evidence for a strong effect of varying the correlation time of the forcing is to be found in Perez Boldyrev 2010, see figure below.



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